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The transformation groupoid of Lorentz invariant space

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Abstract. In a relativistic description of nature, physics may be seen as the transformation of events from one space-like surface in Minkowski space to another such space-like surface. In the present paper an abstract algebra is formulated which characterizes such transformations in much the same sort of way as the algebra of the rotation group characterizes rotational transformations. Both quantum field theory and the classical theory of contact transformations are representations of this algebra. Hence from this viewpoint quantum field theory becomes the group theoretical problem of constructing unitary irreducible representations of the algebra.

1. Introduction

Symmetry groups, of various kinds, play an important role in modern physics and the aim of this paper is to define an algebra of this type which one might expect to be of particular relevance in a relativistic description of nature. In order to explain the motivation behind the construction of this algebra we shall first digress and consider the rotation group. From a physical point of view, the elements of this group algebra are the actual physical operations which one uses to rotate a real body and the composition, or multiplication, law of the algebra is the same as the actual composition attained by successive rotation operations. However, from a more sophisticated viewpoint the rotation group is an abstract algebra, only one of whose representations is the set of physical acts involved in rotating an object. The abstract algebra has other representations, such as the set of substitutional coordinate transformations (often used as a definition) and the unitary matrix representations. The unitary representations also play a role in quantum mechanics and this allows the rotation group to be used in the study of the angular momentum operators. Although this does not lead to any results that cannot be deduced from the mechanics formulae $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, this 'algebra of physical operations' approach does give a different insight into why, for example, the angular momentum operators satisfy the commutation relations that they do. Moreover, this different viewpoint may suggest some extension of the original ideas, such as the introduction of the universal covering group, SU(2).

In the present paper a different algebra of physical operations is constructed, and its unitary representations are related to quantum field theory. Again, the advantage in this is the insight which it offers. The physical operations involved in this new algebra are those which occur in transforming one three-dimensional space-like surface embedded in a four-dimensional Minkowski space into another such surface. Obviously such an algebra is deeply involved in any relativistic description of nature.

The abstract algebra formed is actually a groupoid algebra. The differences between a groupoid and a group are slight and are discussed in § 2. The axioms defining this groupoid are given in § 3. They are based purely on physical (or geometric) operational considerations: no quantum field theoretical concepts as such are involved. This is actually a Lie groupoid and in § 4 it is analysed in terms of infinitesimal generators. In § 5 it is shown that the axioms of the groupoid demand that these infinitesimal generators have a simple form while in § 6 the Lie algebra of the groupoid is deduced. Finally, in § 7, the classical and quantum field representations of the algebra are discussed.

2. A Brandt groupoid

A Brandt (1927) groupoid is an abstract algebra in which a single composition, or multiplication, operation is defined. The only difference between a groupoid and a group is that the product of two elements is not necessarily defined in a groupoid. For the groupoid of interest in this paper the concepts and techniques of ordinary group theory are still applicable.

Groupoids arise naturally in the theory of mapping, or transforming, a given set of objects, say $\{\sigma\}$, into some other set. Corresponding to the mapping operation which takes the object σ into σ', which we denote by $\sigma \to \sigma'$, one can define a groupoid element $U[\sigma \to \sigma']$. When the transformation $\sigma \to \sigma'$ is followed by $\sigma' \to \sigma''$ the result is equivalent to $\sigma \to \sigma''$ and hence the composition law for the algebra is defined to be

$$U[\sigma \to \sigma']U[\sigma' \to \sigma''] = U[\sigma \to \sigma'']. \tag{2.1}$$

Such an algebra obviously satisfies all the defining axioms of a group with the exception that the multiplication $U[\sigma_1 \to \sigma_2]U[\sigma_3 \to \sigma_4]$ may not be defined: the algebra is a groupoid and not a group.

It might be noted that the order of the factors on the left-hand side of (2.1) is a matter of convention, and the above ordering has been chosen in order to agree with the Heisenberg picture conventions of quantum field theory.

There may be several 'mapping labels', $\sigma \to \sigma'$, for each truly independent element of the groupoid. For example, the orientation of a body may be specified by three angles (θ, ϕ, ψ) and hence the elements of the rotation group may be labelled with the mapping scheme above by identifying σ with the set of angles (θ, ϕ, ψ). The multiplication table (2.1) is then valid for the rotation group also, but now each independent member of the group has many different labels. The independent elements in the rotation group are, of course, the 'rotation operations' and these exist independently of the alignment of the object being rotated, thereby giving the redundancy to the mapping labels. The important point to note is that, under such a labelling scheme for the elements of the groupoid, the intrinsic structure of the theory does not lie in the multiplication table (2.1) but in the specifications which define the redundancy in the labelling system.

3. The elements of the groupoid

3.1. Labelling of the elements

In a flat four-dimensional Minkowski space the coordinates are x^α, $(\alpha = 0, 1, 2, 3)$, and the metric is $g_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$. A three-dimensional surface, σ, may be parametrized by $\boldsymbol{u}$ so that

$$\sigma = \{x^\alpha = x^\alpha(u^1u^2u^3), \alpha = 0, 1, 2, 3\}. \tag{3.1}$$

The function $x^\alpha(\boldsymbol{u})$ are assumed to be smooth and suitably differentiable. These surfaces are restricted to being space-like (so that representations of the algebra will actually exist), ie the unit normal to the surface, $n_\alpha(x)$, must be everywhere time-like

$$n_\alpha g^{\alpha\beta} n_\beta = +1. \tag{3.2}$$

The mapping of interest, $\sigma \to \sigma'$, is then the transformation of the points forming the space-like surface σ into the points forming the space-like surface σ'. This mapping is defined independently of the parametrization used, but it is to be noted that two mappings $\sigma \to \sigma'$ and $\sigma \to \sigma''$ are different, even when σ' and σ'' form the same geometric surface, unless the individual points of σ are in both cases transported to identical new positions. The allowed mappings are assumed to be one-to-one, continuously differentiable, etc.

The elements of the abstract algebra, U, are labelled by these mapping operations,

$$U = U[\sigma \to \sigma']. \tag{3.3}$$

These elements are redundantly labelled and hence (3.3) should not be considered as a functional of the initial and final positions of the points in the surface, but it should be considered as being labelled by the act, or operation, involved in making the transformation $\sigma \to \sigma'$. If all such operations could be uniquely defined in such a way that they could be applied to an arbitrary surface then the multiplication law (2.1) would reduce into a group multiplication table. But this is not the case and the algebra remains a groupoid. Even so there is still a large amount of redundancy in the labelling.

3.2. *The unit element*

Consider the unit element $U[\sigma \to \sigma]$. As labelled this could be a functional of the surface σ. But if the element is a characterization of the operation involved, then one 'feels' that this is the same operation (ie no change) no matter which surface σ is involved. Hence this is defined to be

$$U[\sigma \to \sigma] = 1, \tag{3.4}$$

a single element of the groupoid independent of σ.

3.3. *Local deformation assumptions*

The same considerations apply in the following case: consider two different space-like 3-surfaces, σ_1 and σ_2, which are however coincident over some finite region R. Now suppose that some deformation is made to the coincident part of the surfaces, with the deformation entirely restricted to the region R. 'Clearly' it is the same deformation, ie the same geometric operation, no matter whether one considers the region R as belonging to the surface σ_1 or to the surface σ_2. Hence if the above situation corresponds to either $\sigma_1 \to \sigma'_1$ or to $\sigma_2 \to \sigma'_2$, then it is assumed that

$$U[\sigma_1 \to \sigma'_1] \equiv U[\sigma_2 \to \sigma'_2]. \tag{3.5}$$

A much stronger version of this group axiom is also assumed. In the limit of an infinitesimal local deformation the two surfaces σ_1 and σ_2 need only be tangential in order to define a common region. Equation (3.5) is still assumed valid, ie to show two different labelling schemes for the same independent groupoid element, in this limiting situation also.

3.4. *Inhomogeneous Lorentz transformations*

In a Minkowski space, and in contradistinction to a general Riemann space, it is possible to define the inhomogeneous Lorentz group of operations. These operations may be considered as operations which translate and Lorentz rotate objects situated in that space (rather than as coordinate transformations), and as such may be applied to surfaces which exist in the Minkowski space. 'Obviously' these will be the same operations no matter to which surface they are applied. If L denotes an element of the inhomogeneous Lorentz group, and $\sigma \to L(\sigma)$ denotes the effect of this operation when applied to the surface σ, then it is assumed that

$$U[\sigma \to L(\sigma)] \equiv U(L). \tag{3.6}$$

Hence the groupoid element on the left-hand side of (3.6) is not a functional of the surface σ but depends only on the parameters specifying the inhomogeneous Lorentz group element, L.

This last assumption makes the inhomogeneous Lorentz group a subgroup of the total groupoid.

4. Lie groupoid assumptions

Just as in the theory of ordinary groups, additional assumptions must be introduced in order to define a Lie (or continuous) groupoid. Firstly some generalized concept of 'nearness' between the elements of the abstract algebra must be introduced, and then it is assumed that an infinitesimal change in the parameters labelling an element leads to an infinitesimal change in the actual element. The groupoid element corresponding to an infinitesimal change is then expanded in the form $U = 1 - \mathrm{i}G$, where G is called the infinitesimal generator of the transformation (the factor $-\mathrm{i}$ here has been introduced in order to conform to the usual quantum mechanical conventions). In order to make this expansion the composition operation of 'addition' must be introduced into the algebra, but this operation is only introduced for the purpose of analysing the individual groupoid elements. The addition operation is assumed to have all the usual properties normally associated with addition, including commutativity.

In an infinitesimal geometric mapping operation the point $x^\alpha(\boldsymbol{u})$ of the surface σ is transported by the infinitesimal amount $\Delta^\alpha(\boldsymbol{u})$ to the new position $x^\alpha(\boldsymbol{u})+\Delta^\alpha(\boldsymbol{u})$ on the surface $\sigma+\Delta$, and the groupoid element corresponding to this will be denoted by $U[\sigma \to \sigma+\Delta]$. This shows that an infinitesimal transformation is parametrized by an infinite set, $\{\Delta^\alpha(\boldsymbol{u})\}$, of infinitesimal parameters. The expansion about the unit element then takes the form

$$U[\sigma \to \sigma+\Delta] = 1-\mathrm{i}G[\Delta, \sigma]+\mathrm{O}(\Delta^2) \tag{4.1}$$

where $G[\Delta, \sigma]$ is assumed to be a linear functional of the infinitesimal parameters.

The infinitesimal generator will be written in the form

$$G[\Delta, \sigma] = \int_\sigma J_\alpha(x, [\sigma])\Delta^\alpha(x)\,\mathrm{d}S \tag{4.2}$$

where $J_\sigma(x, [\sigma])$ is an (as yet) arbitrary functional of the surface σ. Here the infinitesimal parameters have been specified by the point x on that surface to which they belong, rather than by the parameters $\boldsymbol{u}$. Although (4.2) is analogous to the corresponding expressions from the theory of ordinary Lie groups, it requires a variety of assumptions in order to make it unique. Firstly, it is an assumption that the measure across the parameters may be taken as being proportional to the surface area element, $\mathrm{d}S$, although it is hard to see what other assumption could be used. Secondly, it is an assumption that $G[\Delta, \sigma]$ is an analytic linear functional of the components $\Delta^\alpha(x)$ and not, for example, of the modulus $|\Delta| = \sqrt{(\Delta^\alpha \Delta_\alpha)}$. Next, it should be realized that (4.2) is not the most general linear functional possible as one may, for example, add linear combinations of surface differentials of $\Delta^\alpha(x)$ to the above expression. However, if it is assumed that $J_\alpha(x, [\sigma])$ may be a generalized function of x, and that there is no need to worry about the end point contributions in an integration by parts operation, then it is possible to transform these alternative forms into (4.2). Throughout the ensuing analysis it is always assumed that in any integration by parts, with the integration being carried out over an infinite space-like surface, that there are no end point contributions. This puts restrictions on any representation of the algebra. All such assumptions as are necessary to ensure the validity of (4.2) will now be assumed without further discussion.

5. The energy–momentum tensor

The dependence of the functional $J_\alpha(x, [\sigma])$ on the surface σ may be completely determined from the group axioms given in § 3.

The locality assumptions of § 3.3 immediately eliminate most of the arbitrariness from the functional form. The elements of the groupoid have been assumed to be independent of the parametrization and to only depend on the geometric properties of the transformation. Hence $J_\alpha(x, [\sigma])$ must be a functional of only geometric properties of the surface, such as the unit normal components, or the surface curvature components, etc. But the locality requirement demands that this must be the same operator for any two tangential surfaces which meet only at the point x. The only geometric quantity which an arbitrary two such surfaces have in common is the unit normal, $n_\beta(x)$, at the point x. Hence $J_\alpha(x, [\sigma])$ is at most an ordinary function of the position x^α and of the unit normal n_β:

$$J_\alpha(x, [\sigma]) \equiv J_\alpha(x, n). \tag{5.1}$$

The inhomogeneous Lorentz group assumptions of § 3.4 then allow the actual functional dependence of $J_\alpha(x, n)$ with respect to the normal n_β to be determined. Evaluating $G[\Delta, \sigma]$ with a constant Δ^α, and equating the result to $\Delta^\alpha P_\alpha$, shows that the generator of a translation in the inhomogeneous Lorentz group is

$$P_\alpha = \int_\sigma J_\alpha(x, n)\,\mathrm{d}S. \tag{5.2}$$

The Lorentz group axioms demand that this operator must be independent of the surface σ. Varying the position of the surface slightly (see appendix) shows that the function $J_\alpha(x, n)$ must satisfy

$$0 = \frac{\partial}{\partial x^r}\frac{\partial}{\partial n_r}J_\alpha(x, n) + n^s\frac{\partial}{\partial x^s}\left(J_\alpha(x, n) - n_r\frac{\partial}{\partial x^r}J_\alpha(x, n)\right)$$
$$+ R_{st}\left[\frac{\partial}{\partial n_s}\frac{\partial}{\partial n_t}J_\alpha(x, n) + g^{st}\left(J_\alpha(x, n) - n_r\frac{\partial}{\partial x^r}J_\alpha(x, n)\right)\right]. \tag{5.3}$$

In this equation the four components of n_α have been treated as independent variables, but it must be remembered that $J_\alpha(x, n)$ need only be evaluated in the situation where these components satisfy

$$n_\alpha g^{\alpha\beta} n_\beta = +1. \tag{5.4}$$

The quantities R_{st} which appear in equation (5.3) are the components of the surface curvature tensor at the point x (see appendix). This tensor is symmetric and satisfies

$$n^s R_{st} = R_{ts} n^s = 0. \tag{5.5}$$

As equation (5.3) must hold for an arbitrary surface which may have arbitrary values of R_{st}, provided that (5.5) is satisfied, $J_\alpha(x, n)$ must satisfy

$$\frac{\partial}{\partial n_s}\frac{\partial}{\partial n_t}J_\alpha(x, n) + g^{st}\left(J_\alpha(x, n) - n_r\frac{\partial}{\partial x^r}J_\alpha(x, n)\right) = n^s B^t_\alpha(x, n) + n^t B^s_\alpha(x, n), \tag{5.6}$$

where $B^t_\alpha(x, n)$ are arbitrary functions of x and n. In order to determine the functional form of $J_\alpha(x, n)$ the series expansion

$$J_\alpha(x, n) = T_\alpha(x) + T^\beta_\alpha(x) n_\beta + \frac{1}{2!}T^{\beta\gamma}_\alpha(x) n_\beta n_\gamma + \ldots \tag{5.7}$$

will be substituted into (5.6) and the coefficients of the powers of n equated. The author has a proof which does not involve introducing this series expansion, but it is long and inelegant and will not be presented here. Making the corresponding expansion for the functions

$$B^t_\alpha(x, n) = b^t_\alpha(x) + b^{tu}_\alpha(x) n_u + \ldots \tag{5.8}$$

and substituting, gives

$$(T^{st}_\alpha(x) + T^{stu}_\alpha(x) n_u + \ldots) + g^{st}\left[T_\alpha(x) + 0 + \left(\frac{1}{3!} - \frac{1}{2!}\right)T^{\beta\gamma}_\alpha(x) n_\beta n_\gamma + \ldots\right]$$
$$= n^s(b^t_\alpha(x) + b^{tu}_\alpha(x) n_u + \ldots) + n^t(b^s_\alpha(x) + b^{su}_\alpha(x) n_u + \ldots). \tag{5.9}$$

It is to be noted that the linear term $T^\beta_\alpha(x)$ does not appear in this equation and hence cannot be determined from it. Equating the coefficients of the powers of n then gives a series of equations for the coefficients. We shall only consider in detail a single example: the coefficients of the linear powers of n_u in (5.9) must satisfy

$$T^{stu}_\alpha(x) = g^{su} b^t_\alpha(x) + g^{tu} b^s_\alpha(x). \tag{5.10}$$

At this stage it must be remembered that a solution is required only in the situation where (5.4) holds. But when the normal is truly a unit vector, (5.10) shows that the third-order contribution of the series expansion for $J_\alpha(x, n)$ is

$$\frac{1}{3!}T^{stu}_\alpha(x)n_s n_t n_u \equiv \frac{1}{3}b^\beta_\alpha(x)n_\beta, \tag{5.11}$$

and this has the same functional form as the contribution to $J_\alpha(x, n)$ from the term $T^\beta_\alpha(x)n_\beta$. The same type of consideration may be applied to every order of (5.9) and it leads to the final conclusion that the functional form of $J_\alpha(x, n)$ (when (5.4) holds) must be equivalent to

$$J_\alpha(x, n) \equiv T^\beta_\alpha(x)n_\beta. \tag{5.12}$$

The quantities $T^\beta_\alpha(x)$ are not completely arbitrary. Substituting (5.12) into (5.3) shows that these quantities must also satisfy the conservation condition

$$\frac{\partial}{\partial x^\beta}T^\beta_\alpha(x) \equiv 0. \tag{5.13}$$

A similar analysis can be made with respect to the Lorentz rotational operators, which follow from evaluating $G[\Delta, \sigma]$ with

$$\Delta^\alpha(x) = \omega^\alpha{}_\beta(x^\beta - x^\beta_0) \tag{5.14}$$

where $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ is infinitesimal. This analysis will not be given in detail but it leads to the result that $T_{\alpha\beta}(x)$ must be symmetric with respect to interchanging α and β, ie

$$T_{\alpha\beta}(x) \equiv T_{\beta\alpha}(x). \tag{5.15}$$

The multiplication law, (2.1), for the groupoid implies that

$$U[L(\sigma) \to L(\sigma')] = U^{-1}(L)U[\sigma \to \sigma']U(L), \tag{5.16}$$

and this in turn implies that the quantities $T^\beta_\alpha(x)$ must transform as tensor quantities under the action of the Lorentz rotational elements, $U(L)$, of the groupoid.

Hence we have shown that the defining axioms of the abstract groupoid imply that the infinitesimal generator must have the form

$$G[\Delta, \sigma] = \int_\sigma \Delta^\alpha(x)T^\beta_\alpha(x)n_\beta\, dS = \int_\sigma \Delta^\alpha(x)T^\beta_\alpha(x)\, dS_\beta \tag{5.17}$$

where $T^\beta_\alpha(x)$ is a symmetric, conserved tensor operator. We shall call $T^\beta_\alpha(x)$ the energy–momentum tensor, as it may be identified as such in quantum field theory. That the form (5.17) for the infinitesimal generator is a direct consequence of the purely physical considerations embodied in the axioms of § 3 is the most important result of this paper.

6. The Lie algebra of the groupoid

The Lie algebra of a Lie group provides a set of commutation relations which must be satisfied by the infinitesimal generators of the group. In the present groupoid a similar algebra exists in the form of commutation relations for the energy–momentum tensors.

The Lie algebra of the groupoid expresses the integrability conditions on the infinitesimal generators. If an infinitesimal transformation of a surface, as described by the parameters $\{\Delta^\alpha_1(x)\}$, is followed by a second infinitesimal transformation, as given by the parameters $\{\Delta^\alpha_2(x)\}$, then the result is a single groupoid element corresponding to the combined transformation. But the combined transformation is independent of the order of the above two sets of infinitesimal transformations. Hence the infinitesimal generators must be such that they satisfy

$$(1-\mathrm{i}G[\Delta_1,\sigma])\{1-\mathrm{i}(G[\Delta_2,\sigma]+\delta_1 G[\Delta_2,\sigma])\} = (1-\mathrm{i}G[\Delta_2,\sigma])\{1-\mathrm{i}(G[\Delta_1,\sigma] + \delta_2 G[\Delta_1,\sigma])\} \tag{6.1}$$

where $\delta_1 G[\Delta_2,\sigma]$ is the change in the infinitesimal generator $G[\Delta_2,\sigma]$ on shifting the surface σ by the amounts $\{\Delta_1^\alpha(x)\}$. Expanding this shows that the integrability condition on the infinitesimal generators is

$$\frac{1}{\mathrm{i}}[G[\Delta_1,\sigma], G[\Delta_2,\sigma]]_- = \delta_2 G[\Delta_1,\sigma]-\delta_1 G[\Delta_2,\sigma] \tag{6.2}$$

where $[A, B]_- \equiv AB-BA$ denotes the commutator of the two enclosed quantities.

Let us consider in more detail the change in

$$G[\Delta_1,\sigma] = \int_\sigma \Delta_1^\alpha(x) T_\alpha^\beta(x)\,\mathrm{d}S_\beta \tag{6.3}$$

when the surface σ is altered by the parameters $\{\Delta_2^\alpha(x)\}$. The change in the energy–momentum tensor is

$$\delta_2 T_\alpha^\beta(x) = \frac{\partial T_\alpha^\beta(x)}{\partial x^a}\Delta_2^a(x), \tag{6.4}$$

while the change in the surface area element (see appendix) is

$$\delta_2(\mathrm{d}S_\beta) = \mathrm{d}S\, D_{\beta\alpha}(x)\Delta_2^\alpha(x). \tag{6.5}$$

Here $D_{\beta a}(x)$ is the surface differential operator at the point x,

$$D_{\beta a}(x) \equiv n_\beta(x)\frac{\partial}{\partial x^a}-n_a(x)\frac{\partial}{\partial x^\beta}, \tag{6.6}$$

and, despite the notation, this differential operator acts only on surface values of a function. Using these results

$$\delta_2 G[\Delta_1,\sigma] = \int_\sigma \left(\Delta_1^\alpha(x)\frac{\partial T_\alpha^\beta(x)}{\partial x^a}\Delta_2^a(x)\right)\mathrm{d}S_\beta + \int_\sigma (\Delta_1^\alpha(x)T_\alpha^\beta(x)D_{\beta a}(x)\Delta_2^a(x))\,\mathrm{d}S. \tag{6.7}$$

This result may be simplified by integrating by parts (and ignoring any contributions from the end points at infinity, see § 4) with the formula

$$\int_\sigma (\Delta_1^\alpha(x)T_\alpha^\beta(x)D_{\beta a}(x)\Delta_2^a(x))\,\mathrm{d}S = -\int_\sigma (\Delta_2^a(x)D_{\beta a}(x)\Delta_1^\alpha(x)T_\alpha^\beta(x))\,\mathrm{d}S. \tag{6.8}$$

Combining, and using the conservation condition (5.13), then gives

$$\delta_2 G[\Delta_1,\sigma] = -\int_\sigma \mathrm{d}S(y)(\Delta_2^a(y)T_\alpha^\beta(y)D_{\beta a}(y)\Delta_1^\alpha(y)). \tag{6.9}$$

In order to deduce the commutation relations for the energy–momentum tensors from (6.2), it is convenient to express the above result as a double surface integral. This may be achieved by introducing the surface delta function $\delta_s(x, y)$ (see appendix). This even function is defined only on a surface running through the two points x and y; it is zero except when the point x is in the neighbourhood of the point y; and it satisfies

$$\int_\sigma dS(x) f(x)\delta_s(x, y) = f(y) \tag{6.10}$$

for an arbitrary 'good' function defined on the surface σ. With the aid of this generalized function (6.9) may be written as

$$\delta_2 G[\Delta_1, \sigma] = \int_\sigma dS(x) \int_\sigma dS(y) \Delta_1^\alpha(x) \Delta_2^a(y)(-T_\alpha^\beta(y) D_{\beta a}(y) \delta_s(y, x)). \tag{6.11}$$

The quantity $\delta_1 G[\Delta_2, \sigma]$ may be expressed in a similar double integration form, as may the left-hand side of (6.2). As the integrability condition (6.2) must be valid for arbitrary sets of infinitesimal parameters, it implies that the Lie algebra for the groupoid is

$$\frac{1}{i}[T_\alpha^\beta(x) n_\beta(x), T_a^b(y) n_b(y)]_- = T_a^b(x) D_{ba}(x) \delta_s(x, y) - T_\alpha^\beta(y) D_{\beta a}(y) \delta_s(y, x). \tag{6.12}$$

To be pedantic this is not strictly a 'Lie algebra' (and should perhaps be referred to as a 'Lie algebroid') as the commutation related needs hold only for points x, y which have a space-like separation between them. It must also hold for any space-like surface constructed such that these two points lie in it: the normals and the surface delta function are defined once the surface is given. As (6.12) implies that $T_\alpha^\beta(x)$ commutes with $T_a^b(y)$ for finite separation distances between x and y, this surface needs only be constructed when the two points are in the same neighbourhood.

The problem of finding representations of the total groupoid algebra has now been reduced to the problem of finding symmetric (5.15), conserved (5.13), quantities $T_\alpha^\beta(x)$ which provide a representation of the Lie algebra (6.12). The algebra itself implies the conventional Lie algebra for the generators of the inhomogeneous Lorentz group, and that the quantities $T_\alpha^\beta(x)$ must transform as tensor quantities under the action of these elements of the groupoid.

7. Conclusion

This paper has attempted to define, and to deduce the Lie algebra of, a particular abstract algebra of physical relevance. The defining axioms of this groupoid algebra have been correctly chosen, as far as this paper is concerned, if the algebra may be realized in terms of those physical operations which are involved in the alteration of given space-like 3-surfaces (embedded in a Minkowski space) into other such surfaces, and if the axioms fully capture the 'essence' of such transformations. Hence the paper has succeeded (or not) in its aim depending on just how well the defining axioms given in § 3 measure up to these criteria, and this must be adjudicated upon by the individual readers. On the other hand the usefulness of the groupoid depends on the existence of other representations of the algebra, and the insight which is provided by correlating the properties of these other representations with the properties of this algebra of physical operations. By way of a conclusion to the paper, two other such representations will now be discussed.

The canonical theory of 'contact transformations' provides one representation of the groupoid. Consider (for simplicity) a scalar field $\phi(x)$ which makes

$$S = \int_{\sigma_0}^{\sigma} \mathscr{L}\left(\phi(x), \frac{\partial\phi(x)}{\partial x^\alpha}\right) \mathrm{d}^4 x \tag{7.1}$$

an extremal. The resultant Euler–Lagrange differential equation may be seen as a transformation device which takes the corresponding Cauchy data as given on one surface into an equivalent set of Cauchy data on a different surface. In the theory of contact transformations this transporting of the Cauchy data is derived via the action functional, $S[\phi_0\sigma_0|\phi\sigma]$. This is a functional of the field values $\{\phi(x(\boldsymbol{u}))\}$ on the two surfaces, and the numerical value of this functional is equal to the numerical value of (7.1),

$$S[\phi_0\sigma_0|\phi\sigma] = S, \tag{7.2}$$

when it is evaluated with an extremal solution which has the values $\{\phi_0(\boldsymbol{u})\}$ and $\{\phi(\boldsymbol{u})\}$ on the two surfaces. In the corresponding representation of the groupoid, the groupoid element is

$$U[\sigma_0 \to \sigma] \equiv U[\phi_0\sigma_0|\phi\sigma] = \exp(\mathrm{i}S[\phi_0\sigma_0|\phi\sigma]), \tag{7.3}$$

and the groupoid multiplication operation is defined as ordinary multiplication,

$$U[\phi_0\sigma_0|\phi_2\sigma_2] = U[\phi_0\sigma_0|\phi_1\sigma_1]U[\phi_1\sigma_1|\phi_2\sigma_2], \tag{7.4a}$$

coupled with the criterion that the values of $\{\phi_1\}$ in the right-hand side of (7.4*a*) must be such that the surface functional differential equation

$$\frac{\delta}{\delta\phi_1(x_1)} U[\phi_0\sigma_0|\phi_2\sigma_2]|_{\{\phi_0\},\{\phi_2\}} \equiv \frac{\delta}{\delta\phi_1(x_1)}(U[\phi_0\sigma_0|\phi_1\sigma_1]U[\phi_1\sigma_1|\phi_2\sigma_2]) = 0 \tag{7.4b}$$

is satisfied. It is clear from the general theory of contact transformations that these elements, with the above multiplication law, satisfy the groupoid multiplication requirement (2.1).

The infinitesimal element of the algebra must be such that when it multiplies

$$U[\phi_0\sigma_0|\phi\sigma] \text{ it gives } U[\phi_0\sigma_0|\phi\sigma+\Delta],$$

ie it is

$$(1-\mathrm{i}G[\Delta,\sigma]) = \left(1-\mathrm{i}\int_\sigma \Delta^\alpha(x) T^\beta_\alpha(x)\,\mathrm{d}S_\beta\right) \tag{7.5}$$

with

$$T^\beta_\alpha(x) = \frac{\partial\phi}{\partial x^\alpha}\frac{\partial\mathscr{L}}{\partial(\partial\phi/\partial x^\beta)} - \delta^\beta_\alpha\mathscr{L}. \tag{7.6}$$

This is the usual form of the energy–momentum tensor for a classical field and the infinitesimal generator is the same one which would normally appear in a Poisson bracket formalism. Poisson brackets enter into the present formalism from a consideration of the groupoid product of two infinitesimal generators. The multiplication law (7.4) is not directly applicable to the product of two infinitesimal generators on the same space-like surface, but making two successive infinitesimal transformations and equating the result to the left-hand side of (6.1) shows that

$$G[\Delta_1, \sigma] \times G[\Delta_2, \sigma] \equiv G[\Delta_1, \sigma]G[\Delta_2, \sigma] - \mathrm{i} \int_\sigma \frac{\delta G[\Delta_1, \sigma]}{\delta \pi(x)} \frac{\delta G[\Delta_2, \sigma]}{\delta \phi(x)} \, \mathrm{d}S, \tag{7.7}$$

where x denotes the groupoid product and the multiplications on the right-hand side are ordinary numeric multiplications. In this formula

$$\pi(x) = n_\alpha \frac{\partial \mathscr{L}}{\partial(\partial \phi / \partial x^\alpha)} \tag{7.8}$$

is the conjugate momentum of the field with respect to the surface σ. Consequently the groupoid commutator bracket is

$$[G[\Delta_1, \sigma], G[\Delta_2, \sigma]]_- \equiv \mathrm{i}\{G[\Delta_1, \sigma], G[\Delta_2, \sigma]\}_\mathrm{P}, \tag{7.9}$$

where $\{A, B\}_\mathrm{P}$ is the traditional Poisson bracket. In this representation the Lie algebra of the groupoid becomes

$$\{T^\beta_\alpha(x)n_\beta, T^b_a(y)n_b\}_\mathrm{P} = T^b_a(x)D_{b\alpha}(x)\delta_s(x, y) - T^\beta_\alpha(y)D_{\beta a}(y)\delta_s(y, x) \tag{7.10}$$

and the validity of this equation may be checked by directly substituting (7.6).

Relativistic quantum field theory provides the second representation of the abstract groupoid algebra. In this representation the groupoid elements are unitary matrix operators defined with respect to a Hilbert space, and the groupoid multiplication operation is defined as matrix multiplication. In quantum field theory the basic Lie algebra of the groupoid was first discovered as a commutation relation for the energy–momentum tensors by Schwinger (1962), who later gave a general derivation of the relation by studying the response of an arbitrary system to a perturbation by an external gravitational field (Schwinger 1963a). Schwinger states that this commutation relation is 'the most fundamental equation of relativistic quantum field theory'. The groupoid algebra constructed here gives a new insight into the origin of this commutation relation.

It is perhaps worth stressing that the underlying metaphysics behind the present groupoid approach is quite different from that which underlies Schwinger's (1951, 1953) variational assumption. Moreover, the content of the two theories is not the same. Indeed, as far as local field operator solutions are concerned, the variational assumption appears to be more general than the groupoid algebra approach. Schwinger (1963b) has shown that the basic commutation relations (6.12) are not satisfied by fields which describe spin $\frac{3}{2}$ (or higher) particles and that in these cases the variational assumption leads to a different set of commutation relations. Within the spirit of the groupoid axioms, it is not easy to see how the present formalism could be altered so as to lead to these modified commutation relations. In the groupoid formalism such particles would have to be composite and not elementary.

From the point of view of the groupoid algebra, the problem of constructing a relativistic quantum field theory is synonymous with the problem of constructing a unitary irreducible representation of the Lie algebra (6.12). The space covered by an irreducible representation of the inhomogeneous Lorentz subgroup of the total groupoid is then to be interpreted as the space belonging to a single 'particle'. It is of interest to note that no representation of the total groupoid algebra is covered by the space of a single particle and that any such representation involves an indefinite number of particles. At present the only known methods for constructing such irrepresentations of the algebra are based on the introduction of Lagrangian functions†, but from the groupoid point of view it is the Lie algebra, and not the Lagrangian, which is fundamental. It is possible, even probable, that all irrepresentations of the Lie algebra of the groupoid are equivalent to local field solutions of the normal Lagrangian methods, but whether or not this is necessarily the case is unknown to the present author.

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Appendix. Some surface formulae

When the points of the 3-surface are parametrized as $x^{\alpha}(\boldsymbol{u})$ the surface area element is given by

$$dS_{\alpha} = \epsilon_{\alpha\beta\gamma\delta}\frac{dx^{\beta}}{du^1}\frac{dx^{\gamma}}{du^2}\frac{dx^{\delta}}{du^3}du^1\,du^2\,du^3 \tag{A.1a}$$

$$= n_{\alpha}J(\boldsymbol{u})\,d^3u = n_{\alpha}\,dS, \tag{A.1b}$$

where n_{α} is the unit normal and dS is the magnitude of the surface area element. In terms of these quantities the surface delta function may be defined as

$$\delta_s(x, y) = \frac{1}{J(\boldsymbol{u})}\delta^{(3)}(\boldsymbol{u}-\boldsymbol{u}^1) \tag{A.2}$$

where $\boldsymbol{u}$ parametrizes the point x, and $\boldsymbol{u}^1$ the point y. This is the quantity introduced at equation (6.10).

With the aid of the surface delta function, the surface functional derivative $\delta/\delta\phi(y)$ of a surface functional $F[\{\phi(x)\}, \sigma]$ of the surface values $\phi(x(\boldsymbol{u}))$ is defined by the expansion

$$F[\{\phi(x)+\epsilon\delta_s(x, y)\}, \sigma] = F[\{\phi(x)\}, \sigma]+\epsilon\frac{\delta F}{\delta\phi(y)}+O(\epsilon^2). \tag{A.3}$$

This is the functional derivative which appears in the Poisson bracket (7.10).

† The 'representation argument' would be: if the infinitesimal generator has the same functional form as in some classical representation, but is now constructed from local field operators which satisfy the same Euler–Lagrange differential equations *and* the operator boundary condition $i^{-1}[\phi(x), \pi(y)]_{\pm} = \delta_s(x, y)$, then (apart from ordering problems) it is an algebraic consequence of the boundary conditions that

$$i^{-1}[G_1, G_2]_- \equiv \{G_1, G_2\}_P$$

and it is a consequence of the equations of motion that (7.10) is satisfied. Hence such a generato provides an operator representation of the Lie algebra of the groupoid.

The surface differential operator $D_{\alpha\beta}(x)$ of equation (6.6), acts on surface functions and not functionals. This antisymmetric operator is identically zero unless one of its tensor components sustains the unit normal to the surface. Hence one also defines

$$D_\alpha \equiv n^\beta D_{\beta\alpha}(x) = \frac{\partial}{\partial x^\alpha} - n_\alpha n^\gamma \frac{\partial}{\partial x^\gamma}. \tag{A.4}$$

This reduced operator satisfies

$$n^\alpha D_\alpha \equiv 0 \tag{A.5a}$$

and

$$\frac{dx^\alpha}{du^n} D_\alpha \equiv \frac{d}{du^n}, \tag{A.5b}$$

thereby showing that only surface values are involved in the application of these surface operators.

The surface differential operator $D_{\alpha\beta}(x)$ satisfies (when the end point at infinity contributions are ignored) a simple integration by parts formula (cf equation (6.8)). But this implies that the corresponding formula for D_α is

$$\int_\sigma f_1 D_\alpha f_2 \, dS = -\int_\sigma f_2 D_\alpha f_1 \, dS + \int_\sigma f_1 f_2 R \, dS \tag{A.6}$$

where $R = R(x)$ is the mean curvature of the surface at the point x. The mean curvature is defined as

$$R = g^{st} R_{st} \tag{A.7}$$

where

$$R_{st} = D_s n_t \tag{A.8}$$

is the second-order curvature tensor, or the 'second fundamental form', of the surface. This is the quantity which appears in equation (5.3). It satisfies (5.5) and is related to the Riemann curvature tensor of the surface (in this flat enveloping space situation) by

$$R_{\alpha\beta,\gamma\delta} = R_{\alpha\gamma} R_{\beta\delta} - R_{\alpha\delta} R_{\beta\gamma}. \tag{A.9}$$

If the surface is transported by an infinitesimal distance, so that the point $x(\boldsymbol{u})$ shifts to $x(\boldsymbol{u}) + \Delta(\boldsymbol{u})$, then the surface area element alters by

$$\delta(dS_\alpha) = \epsilon_{\alpha\beta\gamma\delta}\left(\frac{d\Delta^\beta}{du^1}\frac{dx^\gamma}{du^2}\frac{dx^\delta}{du^3} + \frac{dx^\beta}{du^1}\frac{d\Delta^\gamma}{du^2}\frac{dx^\delta}{du^3} + \frac{dx^\beta}{du^1}\frac{dx^\gamma}{du^2}\frac{d\Delta^\gamma}{du^3}\right) d^3U \tag{A.10a}$$

$$= dS\,(D_{\alpha\beta}(x)\Delta^\beta(x)). \tag{A.10b}$$

This is equation (6.5). Equation (A.10) immediately implies that

$$\delta(dS) = (D_\alpha \Delta^\alpha)\, dS, \tag{A.11}$$

and

$$\delta(n_\alpha) = -n_\beta(D_\alpha \Delta^\beta). \tag{A.12}$$

Using these results the change in a surface functional on altering the surface may be calculated. For example, the change in the translation generator of the inhomogeneous Lorentz group is

$$\delta P_\alpha = \delta \int_\sigma J_\alpha(x, n)\, \mathrm{d}S \tag{A.13a}$$

$$= \int_\sigma \left(\Delta^\gamma \frac{\partial J_\alpha}{\partial x^\gamma} + \delta n_r \frac{\partial J_\alpha}{\partial n_r} + \frac{\delta(\mathrm{d}S)}{\mathrm{d}S} J_\alpha \right) \mathrm{d}S. \tag{A.13b}$$

Substituting for $\delta(\mathrm{d}S)$ and integrating by parts shows that

$$\int_\sigma \frac{\delta(\mathrm{d}S)}{\mathrm{d}S} J_\alpha\, \mathrm{d}S = \int_\sigma (J_\alpha D_\gamma \Delta^\gamma)\, \mathrm{d}S \tag{A.14a}$$

$$= \int_\sigma (R J_\alpha n_\gamma \Delta^\gamma - \Delta^\gamma D_\gamma J_\alpha)\, \mathrm{d}S \tag{A.14b}$$

$$= \int_\sigma \left(R J_\alpha n_\gamma \Delta^\gamma - \Delta^\gamma \frac{\partial J_\alpha}{\partial x^\gamma} + \Delta^\gamma n_\gamma n^s \frac{\partial J_\alpha}{\partial x^s} - \Delta^\gamma \frac{\partial J_\alpha}{\partial n_t} R_{\gamma t} \right) \mathrm{d}S. \tag{A.14c}$$

Similarly

$$\int_\sigma \delta n_t \frac{\partial J_\alpha}{\partial n_t}\, \mathrm{d}S = \int_\sigma \Bigg[- R n_\gamma n_t \Delta^\gamma \frac{\partial J_\alpha}{\partial n_t} + R_{\gamma t} \Delta^\gamma \frac{\partial J_\alpha}{\partial n_t}$$

$$+ (\Delta^\gamma n_\gamma) \left(\frac{\partial}{\partial x^t} \frac{\partial}{\partial n_t} J_\alpha - n_t n^s \frac{\partial}{\partial x^s} \frac{\partial J_\alpha}{\partial n_t} + R_{st} \frac{\partial^2 J_\alpha}{\partial n_s \partial n_t} \right) \Bigg] \mathrm{d}S. \tag{A.15}$$

Collecting these terms then gives the final result

$$\delta P_\alpha = \int_\sigma (\Delta^\gamma n_\gamma) \Bigg\{ \frac{\partial}{\partial x^t} \frac{\partial}{\partial n_t} J_\alpha + n^s \frac{\partial}{\partial x^s} \left(J_\alpha - n_r \frac{\partial}{\partial n_r} J_\alpha \right)$$

$$+ R_{st} \left[\frac{\partial}{\partial n_s} \frac{\partial}{\partial n_t} J_\alpha + g^{st} \left(J_\alpha - n_r \frac{\partial}{\partial n_r} J_\alpha \right) \right] \Bigg\} \mathrm{d}S, \tag{A.16}$$

and this leads directly to equation (5.3).

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